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Convex Cones and Dentability

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The notion of a dentable subset of a Banach space was introduced by Rieffel [7] in conjunction with a Radon–Nikodym theorem for Banach space-valued measures. Davis and Phelps [1] (also Huff [2]) have shown that those Banach spaces in which Rieffel's Radon–Nikodym theorem is valid are precisely the ones in which every bounded closed convex set is dentable (definition below). Subsequently, Phelps [5] showed that the Banach spaces which have this property (the "Radon–Nikodym property") coincide with those which have the property that every bounded closed convex subset is the closed convex hull of its strongly exposed points (definition below). Saab [8, 9, 10] has extended some of these results to Fréchet spaces. Of immediate interest to us is Saab's characterization of Fréchet spaces with the Radon–Nikodym property as those which have the property that every bounded subset is dentable.

In the present paper, we consider additional geometric characterizations of the Radon–Nikodym property. In particular, we prove that a Banach space has the Radon–Nikodym property if and only if every closed convex cone with a bounded closed base is the closed convex hull of its strongly exposed rays. In Fréchet space, we give a necessary and a sufficient condition in terms of cones for the space to have the Radon–Nikodym property. We wish to thank Professor Phelps for suggesting the construction used in the proof of Theorem 3 which resulted in a shorter proof of that theorem.

DEFINITION 1. Let E be a Hausdorff locally convex space and let E^* denote the topological dual of E.

(i) A subset $C \subset E$ is said to be *dentable* if for every nbhd U of O, there is a point $x \in C$ such that $x \notin$ cl-conv $[C \backslash (x + U)]$, where cl-conv denotes "closed convex hull."

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(ii) A point $x \in C \subset E$ is called a *denting point* of C if for every nbhd U of x, $x \notin \text{cl-conv}(C \setminus U)$.

It follows from the separation theorem for convex sets that x_0 is a denting point of C if and only if for each nbhd U of x_0 there exist $f \in E^*$ and $\alpha \in R$ such that

$$x_0 \in \{x: f(x) < \alpha\} \cap C \subset C \cap U.$$

It is clear that any set whose closed convex hull has a denting point is dentable.

DEFINITION 2. A ray $\rho = \{x + \lambda z: \lambda \geqslant 0, z \neq 0\}$ of a convex set X in a Hausdorff locally convex space is a *denting ray* of X if for any nbhd U of O,

$$\rho' \cap \text{cl-conv}[X' \setminus (x + \langle z \rangle + U)] = \phi,$$

where X' is any bounded convex subset of X, $\rho' = \rho \cap X'$, and $\langle z \rangle$ denotes the one-dimensional subspace generated by z.

THEOREM 1. *Let X be a closed convex cone in a Hausdorff locally convex space E with a bounded closed base Y (that is, there exists $f \in E^*$, $f \neq 0$, such that $Y = \{x: f(x) = 1\} \cap X$ and $X = \{\lambda y: \lambda \geqslant 0, y \in Y\}$) and let $\rho = \{\lambda x_0: \lambda \geqslant 0\}$, where $x_0 \in Y$. Then ρ is a denting ray of X if and only if x_0 is a denting point of Y.*

Proof. It is evident that if ρ is a denting ray of X, then x_0 is a denting point of Y. Conversely, assume that x_0 is a denting point of Y and $Y \neq \{x_0\}$. Let X' be a bounded convex subset of X and let U be a balanced convex nbhd of 0. We may assume $X' \subset \{x: f(x) \leqslant 1\} \cap X$. Since x_0 is a denting point of Y, there exists $g \in E^*$ and $\alpha > 0$ such that

$$x_0 \in \{x: g(x) < \alpha\} \cap Y \subset (x_0 + U) \cap Y.$$

Let $T = \{x: g(x) = \alpha\} \cap Y$. Since $Y \neq \{x_0\}$, we may assume $T \neq \phi$. Now $T + \langle x_0 \rangle$ is a closed convex set, $[0, x_0] = \{\lambda x_0: 0 \leqslant \lambda \leqslant 1\}$ is a compact convex set, and $[0, x_0] \cap (T + \langle x_0 \rangle) = \phi$. Hence, there exist $f_0 \in E^*$ and $\beta > 0$ such that $[0, x_0] \subset \{x: f_0(x) < \beta\}$ and $T + \langle x_0 \rangle \subset \{x: f_0(x) > \beta\}$.

If $y \in Y$ such that $f_0(y) < \beta$, then $f(y) = 1$ and $[x_0, y] \cap T = \phi$. It follows that $g(y) < \alpha$ and hence, $y \in (x_0 + U) \cap Y \subset \langle x_0 \rangle + U$. On the other hand, if $y \in X$ such that $f(y) < 1$ and $f_0(y) < \beta$, then there is a unique $\lambda > 0$ such that $f(y + \lambda x_0) = 1$. From [3, p. 235] we have $y + \lambda x_0 \in X$. Hence, $y + \lambda x_0 \in Y$ and $f_0(y + \lambda x_0) = f_0(y) < \beta$, since $f_0(x_0) = 0$. By the previous argument, it follows that $y + \lambda x_0 \in x_0 + U$ and so $y \in (1 - \lambda) x_0 + U \subset \langle x_0 \rangle + U$. Thus, if $y \in \{x: f_0(x) < \beta\} \cap X'$, then $y \in \langle x_0 \rangle + U$. It follows that $X' \setminus (x_0 + U) \subset \{x: f_0(x) \geqslant \beta\}$. So

$$\rho' \cap \text{cl-conv}[X' \setminus (\langle x_0 \rangle + U)] = \phi,$$

since $f_0(\rho) = 0 < \beta$ and $\rho' = X' \cap \rho$. Therefore, ρ is a denting ray of X.

DEFINITION 3. Let C be a nonempty subset of the Hausdorff locally convex space and let $x \in C$. The point x is a *strongly exposed* point of C if there is an $f \in E^*$ such that $\{C_\alpha\} = \{\{y \in C: f(y) \leqslant \alpha\}: \alpha > f(x)\}$ is a nbhd base of x in C. The functional f is said to *strongly expose* x.

DEFINITION 4 (Zizler [12]). Let X be a nonempty subset of a Hausdorff locally convex space E and let ρ be a closed ray in X. Then ρ is a *strongly exposed ray* of X if there exist $f \in E^*$ and $\alpha \in R$ such that (i) $f(x) = \alpha$ for $x \in \rho$ and $f(x) > \alpha$ for $x \in X \backslash \rho$, and (ii) if U is any nbhd of O and $\{x_i\}$ is a bounded net in X such that $f(x_i) \to \alpha$, then $\{x_i\}$ is eventually in $\rho + U$. The functional f is said to *strongly expose* ρ.

THEOREM 2. *Let X be a closed convex cone in a Hausdorff locally convex space E with a bounded closed base Y and let $\rho = \{\lambda x_0: \lambda \geqslant 0\}$, where $x_0 \in Y$. Then ρ is a strongly exposed ray of X if and only if x_0 is a strongly exposed point of Y.*

Proof. Let $f \in E^*$, $f \neq 0$, such that $\{x: f(x) = 1\} \cap X = Y$ and $X = \{\lambda y: \lambda \geqslant 0, y \in Y\}$. Assume $Y \neq \{x_0\}$. Let $g \in E^*$ such that g strongly exposes x_0 on Y and $g(x_0) < g(y)$ for each $y \in Y \backslash \{x_0\}$. If $g(x_0) \leqslant 0$, then let

$$W = \{x: f(x) > 1\} \cap \{x: g(x) > g(x_0)\}$$

but if $g(x_0) > 0$, then let

$$W = \{x: f(x) < 1\} \cap \{x: g(x) > g(x_0)\}.$$

In either case, W is a nonempty open convex subset of E and $W \cap L = \phi$, where $L = \langle x_0 \rangle$. By the separation theorem there is $f_0 \in E^*$ such that $L \subset \{x: f_0(x) = 0\}$ and $W \subset \{x: f_0(x) > 0\}$ (see [11, Theorem 3.6–E]). Therefore, $f_0(x) > 0$ for $x \in X \backslash \rho$ and ρ is an exposed ray of X.

Let U be a balanced convex nbhd of O. Since x_0 is strongly exposed by g on Y, there is an $\alpha \in R$ such that $g(x_0) < \alpha$ and

$$\{x \in Y: g(x) \leqslant \alpha\} \subset (x_0 + U) \cap Y.$$

Since there are at least two points in Y, α can be chosen so that $\{x \in Y: g(x) = \alpha\} \neq \phi$. Let $z \in Y$ such that $g(z) = \alpha$ and $f_0(z) = \beta > 0$. Then

$$\begin{aligned} \{x: f_0(x) = \beta\} \cap \{x: f(x) = 1\} &= (z - x_0) + \{x: f_0(x) = 0\} \cap \{x: f(x) = 1\} \\ &= (z - x_0) + \{x: g(x) = g(x_0)\} \cap \{x: f(x) = 1\} \\ &= \{x: g(x) = \alpha\} \cap \{x: f(x) = 1\}. \end{aligned}$$

It follows that

$$\{x \in Y: f_0(x) \leqslant \beta\} = \{x \in Y: g(x) \leqslant \alpha\} \subset (x_0 + U) \cap Y.$$

If $\{x_i\}$ is a bounded net in X such that $f_0(x_i) \to 0$, then $f_0[(1/M)\, x_i] \to 0$ and $(1/M)\, U$ is a balanced convex nbhd of O, where $M = \sup\{f(x_i)\}$. Hence, there is a $\beta > 0$ such that

$$\{x \in Y : f_0(x) \leqslant \beta\} \subset [x_0 + (1/M)\, U] \cap Y.$$

Since $f_0[(1/M)\, x_i] \to 0$, there is I so that $f_0[(1/M)\, x_i] \leqslant \beta$, whenever $i > I$. For each $i > I$, there is a $\lambda_i \in [0, 1]$ such that $f[(1/M)\, x_i + \lambda_i x_0] = 1$ and it follows that $(1/M)\, x_i + \lambda_i x_0 \in Y$. Then $f_0[(1/M)\, x_i + \lambda_i x_0] = f_0[(1/M)\, x_i] \leqslant \beta$ for each $i > I$; that is, $(1/M)\, x_i + \lambda_i x_0 \in x_0 + (1/M)\, U$ and $x_i \in M(1 - \lambda_i)\, x_0 + U \subset \rho + U$. Therefore, ρ is a strongly exposed ray of X.

THEOREM 3. *Let C be a nonempty bounded closed convex subset of a Hausdorff locally convex space E such that $0 \notin C$. If the convex cone $X = R^+C = \{\lambda x \colon \lambda \geqslant 0, x \in C\}$ has a strongly exposed ray, then C has a denting point.*

Proof. Choose $f \in E^*$ such that $Y = \{x : f(x) = 1\} \cap X$ is a base for X (for example, f is determined by a hyperplane which separates 0 and C). Let ρ be a strongly exposed ray of X and let $g \in E^*$ such that g strongly exposes ρ (say $g(\rho) = 0$ and $g(x) > 0$ for $x \in X \backslash \rho$). Let $z \in \rho$ such that $z \neq 0$ and $x_0 = \lambda_0 z \in C$, where $\lambda_0 = \sup\{\lambda \colon \lambda z \in C\}$. Assume that $x_0 \in Y$; that is, $f(x_0) = 1$. For each $\beta \in (0, 1)$, let

$$A_\beta = C \cap \{x : f(x) > 1 - \beta\} \cap \{x : g(x) < \beta\}.$$

We will show that if U is any nbhd of O, then there exists $\beta \in (0, 1)$ such that $x_0 \in A_\beta \subset x_0 + U$. It suffices to show that if $\{\beta_i\}$ is a net of decreasing positive real numbers, $\beta_i \to 0$ and $x_i \in A_{\beta_i}$, for each i, then $x_i \to x_0$. Note that $f(x_i) \to 1$. If not, then since f is bounded on C there is a subnet $\{x_j\}$ of $\{x_i\}$ such that $f(x_j) \to t \neq 1$. Since $f(x_j) > 1 - \beta_j$, we must have $t > 1$ and we can assume $f(x_j) > 1$, for all j. Let $y_i = [1/f(x_i)]\, x_i$, then $y_i \in Y$ and $g(y_i) \to 0 = g(x_0)$. Since x_0 is strongly exposed on Y by g, we must have $y_i \to x_0$. But then

$$y_j + (1 - [1/f(x_j)])\, x_0 = [1/f(x_j)]\, x_j + (1 - [1/f(x_j)])\, x_0$$

is in C and converges to $x_0 + [1 - (1/t)]\, x_0 \in C \cap \rho$, contradicting our choice of x_0. Then $f(x_i) \to 1$ and $y_i = [1/f(x_i)]\, x_i$ converges to x_0, which means $x_i = f(x_i)\, y_i$ converges to x_0.

Now suppose $x_0 \in \text{cl-conv}(C \backslash A_\beta)$ for some $\beta > 0$. Since

$$C \backslash A_\beta = [C \cap \{x : g(x) \geqslant \beta\}] \cup [C \cap \{x : f(x) \leqslant 1 - \beta\}],$$

we can choose nets $\{\lambda_i\} \subset [0, 1]$, $\{x_i\} \subset C$ and $\{y_i\} \subset C$ such that $g(x_i) \geqslant \beta$ and $f(y_i) \leqslant 1 - \beta$ and $z_i = \lambda_i x_i + (1 - \lambda_i)\, y_i \to x_0$. Choosing a subnet if necessary, we can assume $\lambda_i \to \lambda \in [0, 1]$. If $\lambda = 0$, then since $\{x_i\}$ is bounded $y_i \to x_0$, contradicting $f(y_i) \leqslant 1 - \beta < 1 = f(x_0)$. Thus, $\lambda > 0$. Since $g(y_i) \geqslant 0$, then

$$\begin{aligned} 0 \leqslant g(x_i) &= (1/\lambda_i)\, g(z_i) - (1/\lambda_i)\,(1 - \lambda_i)\, g(y_i) \\ &\leqslant (1/\lambda_i)\, g(z_i) \to (1/\lambda)\, g(x_0) = 0, \end{aligned}$$

contradicting $g(x_i) \geqslant \beta > 0$. Thus, $x_0 \notin$ cl-conv$(C \backslash A_\beta)$, for each $\beta > 0$. Let U be a nbhd of 0, then there exists $\beta > 0$ such that $x_0 \in A_\beta \subset x_0 + U$ and $x_0 \notin$ cl-conv$(C \backslash A_\beta)$. Therefore, $x_0 \notin$ cl-conv$[C \backslash (x_0 + U)]$ and it follows that x_0 is a denting point of C.

In the proof of Theorem 3, the denting point x_0 is not necessarily a strongly exposed point of C as the following example shows. Let $D_1 = \{(x, y) = (x - 1)^2 + (y - 3)^2 \leqslant 1\}$ and $D_2 = \{(x, y)\colon (x - 1)^2 + (y - 2)^2 \leqslant 1\}$ be discs in the (x, y) plane. Let $C = \text{conv}(D_1 \cup D_2)$ and let X be the cone generated by C with vertex $(0, 0)$. It is easy to see that the nonnegative y-axis is a strongly exposed ray of X which intersects C in the closed segment $\{(0, y) = 2 \leqslant y \leqslant 3\}$. However, neither $(0, 2)$ nor $(0, 3)$ is an exposed point of C.

If C is a bounded subset of a Banach space and the cone generated by cl-conv(C) is the closed convex hull of its strongly exposed rays, then by Theorem 3, C is dentable. On the other hand, if a Banach space has the Radon–Nikodym property, then by Theorem 2, every closed convex cone with a bounded closed base is the closed convex hull of its strongly exposed rays. Therefore, a Banach space has the Radon–Nikodym property if and only if every closed convex cone with a bounded closed base is the closed convex hull of its strongly exposed rays.

Zizler [12] showed that every closed convex weakly locally compact cone in a Banach space is the closed convex hull of its strongly exposed rays. The following example shows that the above characterization of the Radon–Nikodym property can be used to extend this result. Let B be the unit ball in l_1 and let $f = (1, 1, 1,...) \in l^\infty$. Then $H = \{x\colon f(x) = \frac{1}{2}\}$ is a closed hyperplane in l_1 that meets B in a bounded closed convex subset $Y = B \cap H$. Let X be the closed convex cone generated by Y with vertex 0. The Banach space l_1 has the Radon–Nikodym property. Therefore, X is the closed convex hull of its strongly exposed rays. But by [6, Proposition 11.6; 4, Lemma 4.2] one can show that X is not a weakly locally compact cone.

If C is a bounded subset of a Fréchet space and the cone generated by cl-conv(C) is the closed convex hull of its strongly exposed rays, then by Theorem 3, C is dentable. Thus, if in a Fréchet space, every closed convex cone with a bounded closed base is the closed convex hull of its strongly exposed rays, then the space has the Radon–Nikodym property. On the other hand, if a Fréchet space has the Radon–Nikodym property, then by Theorem 1, every closed convex cone with a bounded closed base is the closed convex hull of its denting rays.

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